Econ 802
Lecture Notes on Chapter 3

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This chapter deals with the properties of the profit function. [Note: I will not discuss the envelope theorem or the LeChatelier Principle, but you should read what Varian says about them.]

For this topic, Varian uses the notation

$$\pi(p) \equiv \max_{y \in Y} p \cdot y$$

For the sake of variety, I will assume a single output, use the production function, and use the notation

$$\pi(p, w) \equiv \max_{x \geq 0} \{pf(x) - w \cdot x\}$$

where $p > 0$ is a scalar output price and
$w = (w_1, \ldots w_n) > 0$ is a vector of input prices.

There are four properties a profit function always has. I will state each property and briefly explain why it is true.

(1) Non-decreasing in p and non-increasing in w.

It should be obvious that $\pi(p, w)$ cannot decrease when p rises. The firm always has the option of continuing to use the same input vector x as before. Since $f(x) \geq 0$ and w is not changing, profit cannot fall, and if $f(x) > 0$, it will rise. Of course, the firm may find it optimal to adjust x after p changes, and if so, profit may increase even more.

The second part of the statement is less obvious. Here is a quick proof. Let $w' \geq w^o$ with strict inequality for at least one input i. Let x' be optimal for w' and let x^o be optimal for w^o. We have

$$\underbrace{pf(x') - w'x'}_{\pi(p,w')} \underbrace{\leq}_{\text{because } w' \geq w^o} pf(x') - w^o x' \underbrace{\leq}_{\text{because } x^o \text{ is optimal for } w^o} \underbrace{pf(x^o) - w^o x^o}_{\pi(p,w^o)}$$

This shows that $\pi(p, w') \leq \pi(p, w^o)$.

2 Linearly homogeneous in (p, w)

We need to prove that $\pi(tp, tw) = t\,\pi(p, w)$, for $t > 0$. Let's write this out in detail:

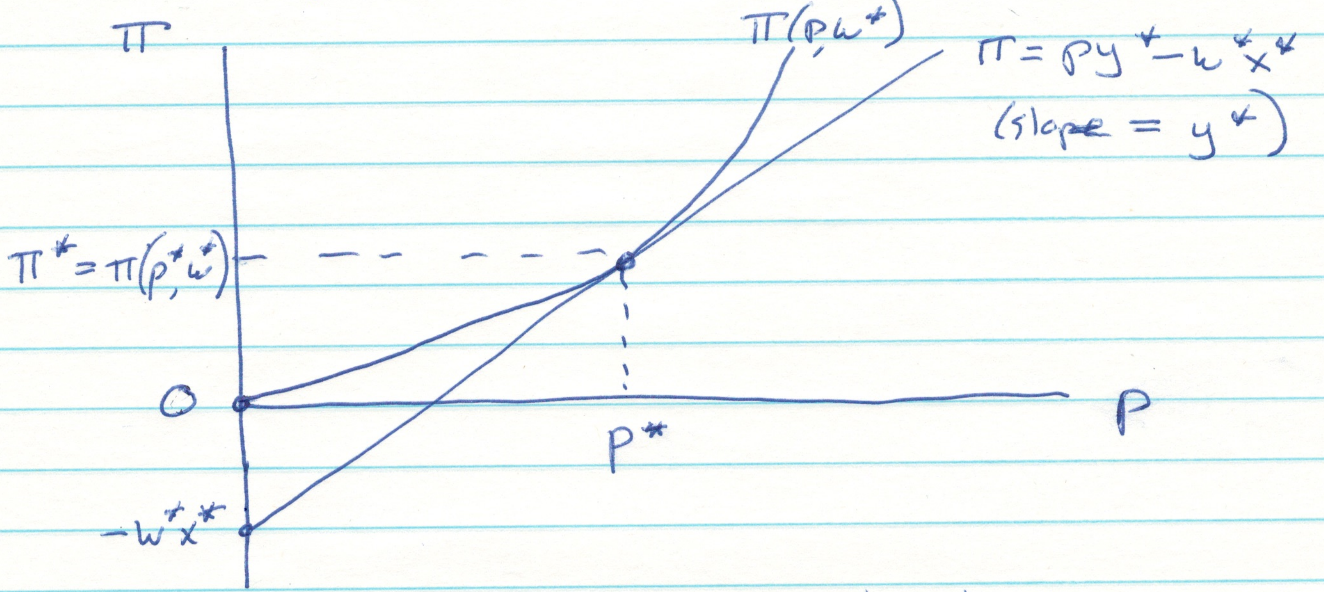
$$\begin{aligned}\pi(tp, tw) &= \max_x \{(tp)f(x) - (tw)x\} \\ &= t \max_x \{pf(x) - wx\} = t\,\pi(p, w).\end{aligned}$$

The key step is the transition from the first line to the second line where we move t outside the max operation. This is justified because the same x is optimal regardless of whether we are maximizing $pf(x) - wx$ or $t\,[pf(x) - wx]$. (it doesn't matter whether we multiply the objective function by a positive constant; whatever was optimal before is still optimal).

3) Convexity in (p, w)

I won't do a full proof, but here is an argument that should clarify what is going on.

Pick some arbitrary prices (p^*, w^*). Let x^* be optimal at these prices and let $y^* = f(x^*)$ be the resulting output. The resulting profit is $\pi^* = p^* y^* - w^* x^*$. Now keep w^* fixed and consider variations in the output price p. One option the firm always has is to stay at (y^*, x^*). If it does, it gets the profit $\pi = p y^* - w^* x^*$, which is linear in p (note that $p = 0$ implies $\pi = -w^* x^* < 0$)



However, when $p \neq p^*$ the firm might be able to increase its profit by re-optimizing (choosing some other x rather than continuing with x^*). Thus the value of the profit function $\pi(p, w^*)$ may be above the straight line shown in the graph. (Note: $\pi(0, w^*) = 0$ because the firm cannot get any revenue when $p = 0$ and sets $x = 0$.) This argument is true both for $p < p^*$ and $p > p^*$ so in general the true function $\pi(p, w^*)$ is above the line, and they meet only at p^*. This is the graphical interpretation of the convexity of $\pi(p, w)$ as a function of the output price. Mathematically we can show that $\pi(p, w)$ is a convex function of the entire price vector (p, w). The proof is not difficult (see Varian).

4. Continuity in (p, w) when $p > 0$, $w > 0$.

The proof comes from the Theorem of the Maximum, which is in chapter 27 of Varian.

Hotelling's Lemma

An important fact about the profit function is that you can differentiate it to obtain output supply and input demand functions (assuming of course that it is differentiable). This involves Hotelling's Lemma, which I will first state and then prove.

Statement of the Lemma:
Let $y(p, w)$ be the firm's output supply function.
Let $x_i(p, w)$ be the firm's (unconditional) demand function for input i.

Then
$$y(p, w) = \frac{\partial \pi(p, w)}{\partial p}$$
$$x_i(p, w) = -\frac{\partial \pi(p, w)}{\partial w_i} \qquad i = 1 \ldots n$$

as long as the derivatives exist and $p > 0$, $w > 0$.

Note: don't forget the negative sign for $x_i(p, w)$! In general $\pi(p, w)$ falls when w_i increases, so if you leave out the negative sign, you will get a negative input demand, which makes no sense in this context (we are using notation where $x \geq 0$ so input levels are non-negative).

Proof of Hotelling's Lemma:

Suppose the prices are (p^*, w^*) and the quantities (y^*, x^*) maximize profit at these prices. Define the function

$$g(p,w) \equiv \pi(p,w) - [py^* - wx^*] \geq 0$$

This is non-negative because for arbitrary prices (p, w), the production plan (y^*, x^*) is not necessarily optimal, and the maximum profit $\pi(p,w)$ is at least as large as what the firm could get from (y^*, x^*).

However we must have $g(p^*, w^*) = 0$ because at these particular prices (y^*, x^*) is optimal, so
$\pi(p^*, w^*) = p^* y^* - w^* x^*$.

This implies that $g(p, w)$ reaches a minimum at (p^*, w^*) and therefore must satisfy the first order conditions for a minimum at these prices. The FOC are

$$\frac{\partial g(p^*, w^*)}{\partial p} = \frac{\partial \pi(p^*, w^*)}{\partial p} - y^* = 0$$

$$\frac{\partial g(p^*, w^*)}{\partial w_i} = \frac{\partial \pi(p^*, w^*)}{\partial w_i} + x_i^* = 0, \quad i = 1 \ldots n$$

Since (p^*, w^*) was chosen arbitrarily, this is true for any prices you want to use and we can remove the stars. The result is

$$y(p,w) = \frac{\partial \pi(p,w)}{\partial p}$$

$$x_i(p,w) = -\frac{\partial \pi(p,w)}{\partial w_i}$$

as claimed.

Hotelling's Lemma is often useful in theoretical work. It can also be useful empirically. For example if you have data on prices and profits (but not quantities) maybe you can estimate the profit function $\pi(p, w)$. If so, you can use Hotelling to compute the output supply and input demand functions that describe the firm's behavior.

Comparative Statics the Easy Way (Duality)

Back in chapter 2, we studied two methods for doing comparative statics with a competitive firm: the FOC method and the algebraic method. Now I will talk about a third method based on Hotelling's Lemma, often called the duality method.

Given Hotelling's Lemma we can write the Hessian of the profit function as follows:

$$\frac{\partial^2 \pi}{\partial (p,w)^2} = \left[\begin{array}{c|ccc} \frac{\partial^2 \pi}{\partial p^2} & \frac{\partial^2 \pi}{\partial p \partial w_1} & \cdots & \frac{\partial^2 \pi}{\partial p \partial w_1} \\ \hline \frac{\partial^2 \pi}{\partial w_1 \partial p} & \frac{\partial^2 \pi}{\partial w_1^2} & \cdots & \\ \vdots & \vdots & & \\ \frac{\partial^2 \pi}{\partial w_n \partial p} & & & \frac{\partial^2 \pi}{\partial w_n^2} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \frac{\partial y}{\partial p} & \frac{\partial y}{\partial w} \\ \hline -\frac{\partial x}{\partial p} & -\frac{\partial x}{\partial w} \end{array}\right]$$

where $\frac{\partial y}{\partial p}$ is a scalar, $\frac{\partial y}{\partial w}$ is $1 \times n$, $-\frac{\partial x}{\partial p}$ is $n \times 1$, and $-\frac{\partial x}{\partial w}$ is $n \times n$

Because $\pi(p,w)$ is convex in (p,w) π has a positive semi-definite Hessian. Furthermore, the Hessian is symmetric.

Therefore, we get symmetric cross-price effects where

$$\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i} \quad \text{for all } i,j$$

$$\frac{\partial y}{\partial w_i} = -\frac{\partial x_i}{\partial p} \quad \text{for all } i = 1 \dots n$$

From pos. semidefiniteness, the diagonal elements must be nonnegative, which implies

$$\frac{\partial y}{\partial p} \geq 0 \quad \text{and} \quad -\frac{\partial x_i}{\partial w_i} \geq 0 \quad \text{for all } i = 1 \dots n$$

$$\text{or} \quad \frac{\partial x_i}{\partial w_i} \leq 0 \quad \text{for all } i.$$

This shows that the firm's output supply curve cannot slope down, and its unconditional input demand curves cannot slope up.

A key point here is that we did not have to invert a matrix to get these results (unlike the method involving FOCs in chapter 2). So the duality method works even if we can't use the implicit function theorem because the sufficient SOCs do not hold. The price we pay is that we don't get equally strong results — we get weak inequalities rather than strict ones. Also note that the algebraic method is more general (it works even without differentiability). But if we <u>can</u> take derivatives the duality approach provides more information.

That's all for chapter 3!